close to the diffracted wave front are clearly qualitatively different.

In Fig.5 we show curves of the pressure distribution on a sphere with a softer damping coating $(\gamma = 5)$ for different instants t. The pressure rise at the point $\varphi = \pi$, due to interaction of waves travelling round the sphere, can prove to be substantial and in some cases is more than twice the incident-wave amplitude.

REFERENCES

1. PEKUROVSKII L.E., PORUCHIKOV V.B. and SOZONENKO YU.A., Interaction of an acoustic wave with a body coated by a thin compressible layer, PMM, 47, 5, 1983.
2. GUZ A.N. and KUBENKO V.D., Theory of non-stationary aero-elastic shells, Methods of claculating shells, 5, Nauk. Dumka, Kiev, 1982.
3. GRIZOLYUK E.I. and GORSHKOV A.G., Non-stationary hydro-elasticity of shells, Sudostroenie, Leningrad, 1974.
4. HÖNL H., MAUE A.W.and WESTPFAHL K., Theorie der Beugung, Handbuch der Physik, 25/1, Springer, Berlin, 1961.
5. ABRAMOWITZ M. and STIGUN A., (Editors), Handbook of mathematical functions with formulas graphs, and mathematical tables, Nat. Bureau of Standards, Washington, 1964.
6. FORSYTHE G.E., MALCOLM M.A. and MOLER C.B., Computer methods for mathematical computations, Englewood Cliffs, 1977.
7. SLEPYAN L.I., Non-stationary elastic waves (Nestatsionarnye uprugie volny), Sudostroenie, Moscow, 1972.
8. SOZONENKO YU.A. (Ed.), Diffraction of acoustic waves by a body with a damping coating, Izd-vo MGU, Moscow, 1985.

Translated by D.E.B.

PMM U.S.S.R.,Vol.51,No.5,pp.656-663,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

STATIONARY VIBRATIONS OF AN ELASTIC HALF-SPACE WITH A CIRCULAR CYLINDRICAL CAVITY SUBJECTED TO A PERIODIC LOAD*

L.A. ALEKSEYEVA

The problem of the stationary vibrations of an elastic half-space with a circular cylindrical cavity subjected to a periodic load along the axis is considered to investigate the state of stress and strain of extended shallow mining shafts under dynamic effects. The problem is reduced to the solution of a system of equations with normal-type determinant by the method of superposition of solutions by using contour integrals of Fourier type and Fourier-Bessel series. The question of the existence and uniqueness of the solution is examined, and its singularities are investigated as a function of the velocity of the moving load or its period. It is shown that Rayleigh surface waves occur in the medium for velocities above the Rayleigh value.

1. Formulation of the problem.

Let us consider an isotropic elastic half-space $x \leqslant h, h > 0$ with Lamé parameters λ, μ, ρ, weakened by a circular cylindrical cavity of radius $R, R < h$ (Fig.1), whose axis OZ is parallel to the half-space boundary. We connect a cylindrical coordinate system (O, r, θ, z) to the cylinder axis, whose polar axis coincides with the OX axis. A load that is stationary in t and periodic in z acts on the cylinder cavity

$$\sigma_{rj} = 2\mu\varepsilon_j p_j(\theta) e^{i(\xi z - \omega t)} \tag{1.1}$$
$$j = r, \theta, z;\ \varepsilon_r = 1,\ \varepsilon_\theta = \varepsilon_z = i$$

*Prikl.Matem.Mekhan.,51,5,836-844,1987

allowing of the Fourier series expansion

$$p_j(\theta) = \sum_n p_j^n e^{in\theta} \tag{1.2}$$

A load allowing of a Fourier transformation in y

$$\sigma_{xj} = \mu\varepsilon_j f_j(y)\, e^{i(\xi z - \omega t)}, \quad j = x, y, z \tag{1.3}$$

$$\varepsilon_x = 1, \varepsilon_y = \varepsilon_z = i, \quad f_j(y) = \int_{-\infty}^{\infty} f_j^*(\eta)\, e^{iy\eta}\, d\eta$$

can also act on the half-space boundary.

Determine the state of stress and strain of the medium.

To solve the problem we use the Lamé potentials Φ, Ψ

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{rot}\Psi \tag{1.4}$$

where $\mathbf{u}$ is the displacement vector. The boundary conditions (1.1) and (1.3) and steady in nature, consequently, the potentials Φ, Ψ have the same dependence on time. We henceforth omit the factor $e^{-i\omega t}$.

We represent Ψ in the cylindrical coordinate system in the form /1/

$$\Psi = \psi_1 \mathbf{e}_z + \operatorname{rot}(\psi_2 \mathbf{e}_z) \tag{1.5}$$

The functions Φ, ψ_1, ψ_2 satisfy the Helmholtz equations /1, 2/

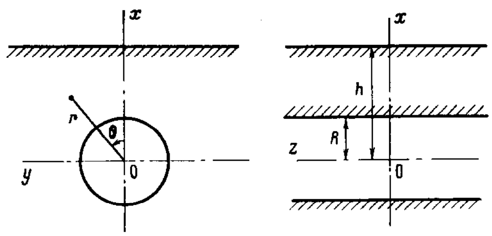

Fig.1

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \alpha_j^2 - \xi^2\right)\varphi_j = 0, \quad j = 0, 1, 2 \tag{1.6}$$

$$(\varphi_0, i\varphi_1, i\varphi_2) = (\Phi, \psi_1, \psi_2); \ \alpha_0 = \alpha = \omega/c_p$$

$$\alpha_1 = \alpha_2 = \beta = \omega/c_s; \ c_p = \sqrt{(\lambda + 2\mu)/\rho}, \ c_s = \sqrt{\mu/\rho}$$

(c_p, c_s are the velocities of propagation of the volume and shear waves).

Using Hooke's law for an isotropic medium

$$\sigma_{ij} = \lambda \sum_k \frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{1.7}$$

and relationships (1.4) and (1.5), we write the boundary conditions for φ_j

$$\left(2\alpha^2 - \beta^2 + 2\frac{\partial^2}{\partial x^2}\right)\varphi_0 + 2i\frac{\partial^2\varphi_1}{\partial x\partial y} - 2\xi\frac{\partial^2\varphi_2}{\partial x^2} = f_x \tag{1.8}$$

$$2\frac{\partial^2}{\partial x\partial y}(\varphi_0 - \xi\varphi_2) + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)i\varphi_1 = if_y$$

$$2i\xi\frac{\partial\varphi_0}{\partial x} - \xi\frac{\partial\varphi_1}{\partial y} - i(2\xi^2 - \beta^2)\frac{\partial^2\varphi_2}{\partial x^2} = if_z \quad \text{for} \quad x = h$$

$$(\xi^2 - {}^1/{}_2\beta^2 + D_1)\,\varphi_0 - iD_2\varphi_1 + \xi(\beta^2 - \xi^2 + D_1)\,\varphi_2 = p_r \tag{1.9}$$

$$-D_2\varphi_0 + i\left(\frac{1}{2}(\beta^2 - \xi^2) + D_1\right) + \xi D_2\varphi_2 = i\rho_\theta$$

$$i\xi\frac{\partial\varphi_0}{\partial r} - \frac{\xi}{2r}\frac{\partial\varphi_1}{\partial\theta} + i\left(\frac{1}{2}\beta^2 - \xi^2\right)\frac{\partial\varphi_2}{\partial r} = ip_z \quad \text{for} \quad r = R$$

$$D_1 = \frac{1}{r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right), \qquad D_2 = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r} - \frac{\partial}{\partial r}\right)$$

We represent φ_j in the form of the superposition of waves emitted by the cavity and the half-space boundaries

$$\varphi_j = \sum_{n=-\infty}^{\infty} a_j^n Z_n(\nu_j, r) e^{in\theta} + \int_L \Lambda(\eta)\, d\eta \tag{1.10}$$

$$\Lambda(\eta) = a_j(\eta) \exp(iy\eta + (x-h)\sqrt{\eta^2 - \nu_j^2}),\ \nu_j = \sqrt{\alpha_j^2 - \xi^2}$$

$$Z_n(\nu_j,\ r) = i\pi (\nu_j/2)^{|n|} H_n^{(1)}(\nu_j r)/\,||n| - 1|!,\ \ \nu_j \neq 0$$

$$Z_0(0,\ r) = \ln r;\ Z_n(0,\ r) = (\operatorname{sgn} n)^n r^{-|n|},\ n \neq 0$$

As is well-known /1-3/, the components of the Fourier-Bessel series in (1.10) are particular solutions of the Helmholtz Eq.(1.6); the Hankel functions $H_n^{(1)}(\nu_j r)$ satisfy the Sommerfeld radiation conditions as $r \to \infty$ if

$$\operatorname{Im} \nu_j \geqslant 0,\ \nu_j \overline{\in} [0, -\infty) \tag{1.11}$$

The damping conditions

$$\operatorname{Im} \sqrt{\eta^2 - \nu_j^2} \leqslant 0, \quad \operatorname{Re} \sqrt{\eta^2 - \nu_j^2} \geqslant 0 \tag{1.12}$$

should be satisfied for the waves emitted by the half-space boundaries.

These conditions impose a constraint on the selection of the contour of integration L and its possible transformations in the plane of complex η: $\eta = \eta_1 + i\eta_2$. To obtain a formal solution of the problem, we assume provisionally that L coincides with the real axis η_1. To give the solution a foundation it is later required that L be transformed because of a number of singularities in the behaviour of the integrands on the real axis η_1.

We also note that for $\xi = \alpha_j$ one or two of the Eqs.(1.6) become Laplace equations. The class of solutions whose first and second derivatives decrease at infinity is described by the functions $Z_n(0, r) e^{in\theta}$. The factor for $H_n^{(1)}(\nu_j r)$ is selected for convenience since it can be shown by starting from the asymptotic form of the Hankel function that /4/

$$\lim_{\nu\to 0} Z_n(\nu, r) = Z_n(0, r), \quad \lim_{\nu\to 0}(Z_0(\nu, r) - \ln \nu) = Z_0(0, r) \tag{1.13}$$

This continuity property permits a number of convenient identical formulas to be obtained for the functions introduced, and is also necessary for investigation of the behaviour of the solution as $\xi \to \alpha_j$.

Therefore, the relationships (1.10) satisfy Eqs.(1.6) (under the assumption that the operation of term-by-term differentiation of the series and functions under the integral sign is valid). The unknown coefficients a_j^n and the functions $a_j(\eta)$ are to be determined.

2. A source periodic in z concentrated on the axis. We assume that a_j^n are known. This is true if a source concentrated on the z axis, those potential can be given by an analogous Fourier-Bessel series, is considered in place of the cavity. To determine $a_j(\eta)$ we go over to a Cartesian coordinate system in relationships (1.10). We use an expansion valid for $x > 0$

$$Z_n(\nu, r) e^{in\theta} = \int_{-\infty}^{\infty} f_n(\eta, \nu) \frac{\exp(iy\eta - x\sqrt{\eta^2 - \nu^2})}{\sqrt{\eta^2 - \nu^2}}\, d\eta \tag{2.1}$$

with the same relationships in the signs of the radicals where

$$f_n(\eta, \nu) = \frac{1}{||n| - 1|!} \left(\frac{\eta + \operatorname{sgn} n \sqrt{\eta^2 - \nu^2}}{2}\right)^{|n|}, \quad |n| + |\nu| \neq 0 \tag{2.2}$$

For $n = 0, \nu = 0$ we will use the formula

$$\ln r = \ln h - \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iy\eta - x|\eta|} - e^{-h|\eta|}}{|\eta|}\, d\eta \tag{2.3}$$

We set $f_0(\eta, 0) = -1/2$. Because of an analogous formula for $H_n(\nu r) e^{in\theta}$ in /5/ formulas (2.1) and (2.3) can be obtained for $\nu = 0$ by a passage to the limit in (1.13). For $\nu = 0$, (2.1) can be obtained differently by using formula 3.384(4) in /6/.

Substituting these relationships into (1.10), we obtain a representation of the potentials in the neighbourhood of the plane boundary $0 < x \leqslant h$ in the Cartesian coordinate system. Later the boundary conditions (1.8) should be used, the terms in $e^{iy\eta}$ should be grouped, and by virtue of the arbitrariness of y the coefficients of $e^{iy\eta}$ on the left- and right-hand sides should be equated. We consequently obtain a system of linear equations in a_j^n and $a_j(\eta)$ which we solve and then obtain

$$a_j(\eta) = \sum_{k=0}^{2} \frac{\Delta_j^k(\eta)}{\Delta_-(\eta)} A_k(\eta) + \frac{\Delta_j(\eta)}{\Delta_-(\eta)} \tag{2.4}$$

$$A_k(\eta) = \frac{\exp(-h\sqrt{\eta^2 - \nu_k^2})}{\nu_1^2\sqrt{\rho^2-\beta^2}} \sum_{n=-\infty}^{\infty} a_k^n f_n(\eta, \nu_k) \tag{2.5}$$

$$\Delta_\pm = 4\rho^2\sqrt{\rho^2-\alpha^2}\sqrt{\rho^2-\beta^2} \pm (2\rho^2-\beta^2)^2$$
$$\Delta_0^0 = \nu_1^2\Delta_+\sqrt{\rho^2-\beta^2}/\sqrt{\rho^2-\alpha^2}$$
$$\Delta_0^1 = 4\nu_1^2\eta\sqrt{\rho^2-\beta^2}(2\rho^2-\beta^2)$$
$$\Delta_0^2 = 4\nu_1^2\xi(\rho^2-\beta^2)(\beta^2-2\rho^2), \quad \Delta_1^0 = 2\beta^2\nu_1^{-2}\Delta_0^1$$
$$\Delta_1^1 = \nu_1^2(2\rho^2-\beta^2)^2 + 4\sqrt{\rho^2-\alpha^2}\sqrt{\rho^2-\beta^2}(\eta^2(\beta^2+\xi^2)-\xi^2\nu_1^2)$$
$$\Delta_1^2 = -8\xi\eta\beta^2(\rho^2-\beta^2)\sqrt{\rho^2-\alpha^2}, \quad \Delta_2^0 = -\nu_1^{-2}\Delta_0^2$$
$$\Delta_2^1 = \beta^{-2}\Delta_1^2, \quad \Delta_2^2 = \Delta_1^1 - 2\nu_1^2(2\rho^2-\beta^2), \quad \rho^2 = \xi^2+\eta^2$$

The components of the loads acting on the plane boundary are

$$\Delta_0 = (\beta^2 - 2\rho^2) f_x^* + 2\sqrt{\rho^2-\beta^2}(\eta f_y^* + \xi f_z^*)$$
$$\nu_1^2\Delta_1 = \beta^2[(2\rho^2-\beta^2) f_y^* - 2\eta\sqrt{\rho^2-\alpha^2} f_x^*] +$$
$$2\xi(2\rho^2-\beta^2-2\sqrt{\rho^2-\alpha^2}\sqrt{\rho^2-\beta^2})(\eta f_z^* - \xi f_y^*)$$
$$\nu_1^2\Delta_2 = \sqrt{\rho^2-\beta^2}[(2\rho^2-\beta^2) f_z^* + 2\xi\sqrt{\rho^2-\alpha^2} f_x^*] +$$
$$\frac{\eta(\eta f_z^* - \xi f_y^*)}{\sqrt{\rho^2-\beta^2}}(4\sqrt{\rho^2-\alpha^2}\sqrt{\rho^2-\beta^2} - 2\rho^2 + \beta^2)$$

It follows from relationships (2.4) that the integrands have singularities in η_1 which depend on ξ. These singularities and the selection of the contour L will be examined in Sect.4. We now assume that the contour L is such that it agrees almost everywhere except in a set of small measure with the axis η_1, the radiation conditions (1.12) are satisfied on L, and the integrands are continuous and twice differentiable with respect to x and y.

3. Diffraction by a hollow cylinder. Resolving system of equations. Let us consider the case $\nu_j \neq 0, j = 0, 1, 2$. To determine a_j^n we use the boundary conditions (1.9). For this we go over to a cylindrical coordinate system in (1.10). It is known /1, 3/ that

$$e^{ikr\cos\zeta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\zeta} \tag{3.1}$$

Relationships (3.1) hold even for complex ζ for which it follows that

$$\exp(iy\eta + x\sqrt{\eta^2-\nu_j^2}) = \sum_{n=-\infty}^{\infty} J_n(\nu_j r) e^{in\theta}\left(\frac{\eta+\sqrt{\eta^2-\nu_j^2}}{\nu_j}\right)^n = \tag{3.2}$$
$$\sum_{n=-\infty}^{\infty} Z_n^0(\nu_j, r) e^{in\theta} f_n(\eta, \nu_j)$$
$$Z_n^0(\nu, r) = \|n| - 1|!\,(2/\nu)^{|n|} J_n(\nu r)$$

Substituting (3.2) into (1.10) we obtain

$$\varphi_j = \sum_n a_j^n Z_n(\nu_j, r) e^{in\theta} + Z_n^0(\nu_j, r) e^{in\theta}\left(\sum_m \sum_{k=0}^{2} a_k^m C_{jm}^{kn} + C_j^n\right) \tag{3.3}$$

Here

$$C_{jm}^{kn} = \int_L \frac{\Delta_j^k(\eta) f_n(\eta, \nu_j) f_m(\eta, \nu_k) E_{kj}(\eta)}{\nu_1^2\sqrt{\rho^2-\beta^2}\Delta_-(\eta)} d\eta \tag{3.4}$$

$$C_j^n = \int_L \frac{\Delta_j(\eta) f_n(\eta, \nu_j) \exp(-h\sqrt{\eta^2 - \nu_1^2})}{\nu_1^2 \sqrt{\rho^2 - \beta^2} \Delta_-(\eta)} d\eta$$
$$E_{kj}(\eta) = \exp(-h(\sqrt{\eta^2 - \nu_j^2} + \sqrt{\eta^2 - \nu_k^2}))$$

Using (3.3) in the boundary conditions (1.9), grouping terms with $e^{in\theta}$ and equating Fourier coefficients on the left- and right-hand sides of the equations, we obtain an infinite system of linear algebraic equations to determine a_j^n

$$\sum_{j=0}^{2} B_k^j(Z_n) a_j^n = -\sum_{j=0}^{2} B_k^j(Z_n^0) \sum_{m=-\infty}^{\infty} \sum_{l=0}^{2} a_l^m C_{jm}^{ln} - \sum_{j=0}^{2} B_k^j(Z_n^0) C_j^n + p_k^n, \quad k = 0, 1, 2, \quad n = 0, \pm 1, \pm 2 \ldots \tag{3.5}$$

The notation in terms of $B_k^j(Z_n)$ is here

$$B_0^0 = (^1/_2(\nu_0^2 R^2 - \xi^2 R^2) - n^2) Z_n(\nu_0, R) + \nu_0 R Z_n'(\nu_0, R)$$
$$B_0^1 = n(\nu_0 R Z_n'(\nu_0, R) - Z_n(\nu_0, R))$$
$$B_0^2 = \nu_0 \xi R^2 Z_n'(\nu_0, R)$$
$$B_1^0 = n(Z_n(\nu_1, R) - \nu_1 R Z_n'(\nu_1 R))$$
$$B_1^1 = (^1/_2 \nu_1^2 R^2 - n^2) Z_n(\nu_1, R) + \nu_1 R Z_n'(\nu_1, R)$$
$$B_1^2 = -n\xi R Z_n(\nu_1, R)$$
$$B_2^2 = \xi R((\nu_1^2 R^2 - n^2) Z_n(\nu_1, R) + \nu_1 R Z_n'(\nu_1, R))$$
$$B_2^1 = n\xi R(Z_n(\nu_1, R) - \nu_1 R Z_n'(\nu_1, R))$$
$$B_2^2 = {}^1/_2 \nu_1 R(\nu_1^2 R^2 - \xi^2 R^2) Z_n'(\nu_1, R)$$
$$J_n'(\nu, R) = dJ(z)/dz\,|_{z=\nu R}$$

as for H_n'; we obtain $B_k^j(Z_n^0)$ by replacing Z_n by Z_n^0.

The formal solution of the problem is completed by solving system (3.5). The displacements and stresses at any point of the medium can be found from the known potentials (1.10) by means of (1.4) and (1.7).

4. On the selection of the contour of integration L. Utilization of the formal Fourier transformation in y on the boundary $x = h$ to solve this problem results in a number of difficulties in principle since the integrands on the real axis, as follows from (2.5), have singularities of the first and even second order pole type, as well as branch points whose location depends on ξ. Therefore, the conditions for the existence of an inverse Fourier transform are not satisfied, and the operation, say, of differentiation under the integral sign is therefore not allowable. However, transformation of the contour L in the neighbourhood of the singular points enables the solution to be represented in the form of contour integrals of Fourier type whose integrands are continuous, differentiable, and satisfy the radiation conditions along L (1.11). Since there are branch points it is necessary to extract the domains of single-valued analytic branches and to construct the contour taking these domains into account.

It follows from (2.5) that the function $a_j(\eta)$ have singularities for

$$\nu_1 = 0, \ \xi = \beta \tag{4.1}$$
$$\rho = \alpha_j, \ \eta = \pm\sqrt{\alpha_j^2 - \xi^2}, \ j = 1, 2 \tag{4.2}$$
$$\Delta_- = 0, \ \eta = \pm\nu_R = \pm\sqrt{\gamma^2 - \xi^2}, \ \gamma = \omega/c_R \tag{4.3}$$

where c_R is the Rayleigh wave velocity in the half-space determined from the solution of (4.3) which has two real roots $\pm\gamma$ /2/ satisfying the condition $\alpha < \beta < \gamma$, $c_R < c_s < c_p$. For periodic loads $\omega / |\xi| = c$ is the propagation velocity along the z axis. If $|\xi| > \gamma$ i.e., $c < c_R$, the integrands have no singularities in η_1 and the real η_1 axis can be taken as the contour L.

For $c = c_R$ we have $\Delta_-(0, \gamma) = \Delta_{-,\eta}(0, \gamma) = 0$. A second-order pole is at the point $\eta = 0$. But since the functions $a_j(\eta)$ are analytic in the neighbourhood of this point and the relationships (2.1) allow transformation of the contour in the neighbourhood of $\eta = 0$, the contour L should bypass this point, say, in the ε-half-plane $0 < \varepsilon < \gamma - \beta$. The direction of traversal is not essential since it does not affect the magnitude of the integral. Let us note that such a transformation of the contour of integration should be allowed by the real load $f_j(y)$ (1.3). This latter is valid for a broad class of loads, particularly finite with finite support.

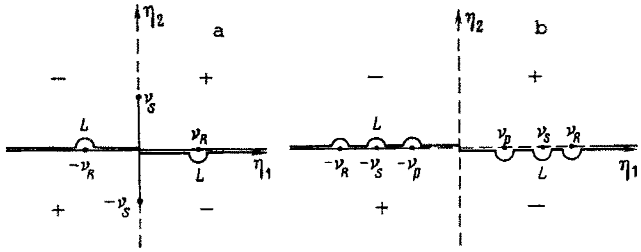

Fig.2

For $c_R < c < c_s$ $(\beta < \xi < \gamma)$ the integrands at the points $\pm\nu_R$ on the η_1 axis have a first-order pole. For this case, a sheet of the Riemann surface of the function $F(\eta, \nu_j) = \sqrt{\eta^2 - \nu_j^2}$, $\operatorname{Re} F \geqslant 0$ with the slits $\eta_1 = 0$, $|\eta_2| > |\nu_j|$ from the branch points $\eta = \pm i|\nu_j|$ on which $\operatorname{Re} F = 0$, is shown in Fig.2a. The signs in the quadrants correspond to the sign of $\operatorname{Im} F$. In this case the value of the integrals depends on the direction of bypassing the singularities $\pm\nu_R$ which should be bypassed in the second and fourth quadrants where conditions (1.12) are satisfied /6/.

As $c \to c_s$ i.e., for $\xi \to \beta$, $\Delta(\eta, \xi) \to \Delta(\eta, \beta) = 0$. It follows from relationships (2.5) that the potentials of the shear waves $\varphi_1, \varphi_2 \to \infty$ while the potential of the volume waves tends to a finite limit.

As $c \to c_s$ and $c \to c_p$, the function $a_j(\eta)$ contains $F(\eta, \nu_j)$ in the denominator, where

$$\lim_{\nu_j \to 0} F(\eta, \nu_j) = \sqrt{\eta^2} = |\eta| \text{ for } \eta = \eta_1$$

We do not succeed in transforming the contour in the neighbourhood of $\eta = 0$ as had been done in the case $c \to c_R$ since by conditions (1.12) $\sqrt{\eta^2} = \eta$ for $\eta_1 > 0, \sqrt{\eta^2} = -\eta$ for $\eta_1 < 0$. Consequently it is impossible to go from the right to the left half-plane with a continuous change in $\sqrt{\eta^2}$ along L without going through $\eta = 0$. At the point $\eta = 0$ the functions have a non-integrable singularity of the type $1/|\eta|$, consequently, even as $c \to c_p$ we obtain $\varphi_0 \to \infty$ the limit of φ_1, φ_2 is finite.

If $c > c_s, c \neq c_p$, besides the first order poles $\eta = \pm\nu_R$ on the axis η_1, integrable singularities $\pm\nu_s = \pm\sqrt{\beta^2 - \xi^2}$, as well as $\pm\nu_p = \pm\sqrt{\alpha^2 - \xi^2}$, appear, if $c > c_p$, which are branch points of the function $F(\eta, \nu_j)$. In Fig.2b where $c > c_p$ the sheet of the Riemann surface of F on which $\operatorname{Re} F \geqslant 0$, is fixed by the slits $\eta_2 = 0$, $|\eta_1| < \nu_s$ and $\eta_1 = 0$. As before, $\operatorname{Im} F \leqslant 0$ in the second and fourth quadrants where the singularities should indeed be bypassed as is shown in Fig.2b.

For the contour selected in such a manner the integrands are continuous and differentiable with respect to x, y on L, the operation of multiple didferentiation under the integral sign is allowable by virtue of the exponential damping of the integrands as $|\eta| \to \infty$ on L and the presence of integrable majorants independent of x, y (Sect.5).

Note that it is not convenient to evaluate the integral (3.4) along L. By using the theory of residues we can represent the integral component in (1.10) in the form (for $c > c_R$):

$$\int_L \Lambda_j(\eta)\, d\eta = \text{V.p.} \int_{-\infty}^{\infty} \Lambda_j(\eta_1)\, d\eta_1 + i\pi \sum_{k=1}^{2} (-1)^k \operatorname{Res} a_j(\eta)|_{\eta=(-1)^k \nu_R} \times$$
$$\exp(iy(-1)^k \nu_R + (x-h)\sqrt{\gamma^2 - \alpha_j^2})$$

The first integral describes the damped waves as $|y| \to \infty$ the second describes Rayleigh waves that occur for $c > c_R$. The same rule should be used when evaluating the coefficients C_{jm}^{kn}, C_j^n (3.4), (3.5).

5. On the solvability of a system of equations. The foundation for the solution of the periodic problem.

The infinite system of Eqs. (3.5) to determine the coefficients a_j^n can be reduced to a system with a determinant of normal type. Let us show this. We introduce the new unknowns

$$b_k^n = \sum_{j=0}^{2} B_k^j (Z_n)\, a_j^n \tag{5.1}$$

We let $D_k^j(Z_n)$ denote a matrix inverse on $B_k^j(Z_n)$. We have

$$b_k^n = \sum_{m=-\infty}^{\infty} \sum_{s=0}^{2} G_{km}^{ns} b_s^m - \sum_{s=0}^{2} B_k^s(Z_n^0) C_s^n + p_k^n \tag{5.2}$$

$$G_{km}^{ns} = \sum_{j=0}^{2} \sum_{l=0}^{2} B_k^j(Z_n^0) D_l^s(Z_m) C_{jm}^{ln}$$

It follows from relationships (2.5) that by virtue of the selection of L the estimates

$$\begin{aligned} &| f_n(\eta, \nu) | < | \nu/2 |^{|n|} / | |n| - 1 |!, \ |\eta| < \nu \\ &| f_n(\eta, \nu) | < | \eta |^{|n|} / | |n| - 1 |!, \ |\eta| > \nu \\ &| \Delta_j^k/\Delta_- | < C, \ \rho < \rho_0; \ | \Delta_j^k/\Delta_- | < C\rho, \ \rho > \rho_0 \end{aligned} \tag{5.3}$$

are valid on L.

Here and below C and ρ_0 are certain positive constants independent of n, m, η. Moreover, for any $\delta, 0 < \delta < \min(2, h - R)$ there exists a $C_\delta > 0$ such that along L

$$| E_{kj}(\eta) | < C_\delta \exp(-2(h - \delta) | \eta |) \tag{5.4}$$

These relationships enable us to evaluate the integrals C_{jm}^{kn} and C_j^n. For example.

$$| C_{jm}^{kn} | < C \frac{| |n| + |m| + 1 |!}{(2h')^{|n|+|m|} | |n| - 1 |! \, | |m| - 1 |!}, \quad h' = h - \delta$$

For sufficiently large $n > N$, $m > M$ the following estimates are true

$$| B_k^j(Z_n^0) | < C |n| R^{|n|}, \ | D_k^j(Z_m) | < C R^{|m|}$$

Consequently

$$| G_{km}^{nl} | < C \frac{|n| + |m| + 1}{| |n| - 2 |! \, | |m| - 1 |!} s^{|n|+|m|}, \quad s = \frac{R}{2h'}$$

Hence

$$\sum_{\substack{|n| > N \\ |m| > M}} | G_{km}^{nl} | < C \sum_{k=0, m=1} \frac{(k+m)!}{(m-1)! \, k!} s^{k+m} = C \sum_{m=1}^{\infty} \left(\frac{s}{1-s} \right)^m = C \frac{s}{1-2s}$$

This last series converges by virtue of the selection of δ, i.e., system (5.2) has a determinant of normal type. The free terms of the system are bounded since $h' > R$ and

$$| C_j^n B_k^j(Z_n^0) | < C |n| (|n| + 1)(R/h')^{|n|}$$

The coefficients $p_k^n \to 0$ as $|n| \to \infty$, by convention. Therefore, the conditions are satisfied for the existence and uniqueness of a bounded solution of system (5.2) that can be found by the method of reduction /7/. Successive approximations, which is equivalent to the method of successive reflections can be used.

The order of decrease of the coefficients b_k^n in n is no worse than the order of p_k^n as $|n| \to \infty$. Consequently, the system of Eqs.(5.1) satisfies the estimate

$$| a_j^n Z_n(\nu_j, r) | < C (R/r)^{|n|} |n|^{-1-s}$$

if $p_k^n = O(|n|^{-s-1})$, $s > 0$ from which the uniform convergence of the Fourier-Bessel series in (1.10) in the domain $r \geqslant R$ follows, as does the inequality

$$| a_j^n f_n(\eta, \nu_j) | < C | R\eta |^{|n|} |n|^{-s} / |n|!$$

Consequently on L

$$| A_j(\eta) | < C \exp(-h' |\eta|) \sum_{|n|} \frac{|R\eta|^{|n|}}{|n|!} < 2Ce^{(R-h')|\eta|}.$$

Therefore, the integrals in (1.10) converge uniformly in x, y in the domain $x \leqslant h$. Analogous uniform estimates can be obtained for the first and second formal derivatives of the series and integrals and their uniform convergence can be shown. The existence and uniqueness of the solution of the problem is shown thereby.

The signs of the radicals were constructed from physical representations in the construction of the solution. Namely, the integrals in (1.10) correspond to the expansion of potentials in plane waves being propagated in the lower half-space and damped at infinity.

Since the functions $a_j(\eta)$ are analytic with a finite number of singularities, the contour can consequently be transformed by going over to equivalent contours. By using the method of stationary phase, it can be shown that for $c < c_R$ relationships (1.10) satisfy the Sommerfeld radiation conditions

$$\varphi_j = O(\sqrt{\rho}),\ \partial\varphi_j/\partial\rho - i\nu_j\varphi_j = o(\sqrt{\rho}). \tag{5.5}$$
$$\rho = \sqrt{(x-h)^2 + y^2}$$

For $c > c_R$ that part of the potentials described by Fourier-Bessel series and the integral in the principal value sense will satisfy conditions (5.5). The components outside the integral describe Rayleigh surface waves of constant amplitude for $x = \text{const}$ independently of $y,\ z$.

It is interesting to compare the solution obtained with an analogous solution in the case of plane deformation. As is shown in /6, 8/, for plane deformation the stationary load on a circular cavity generates Rayleigh waves in a half-plane (this is also clear from the solution presented above), which corresponds to the case $\xi = 0\ (c = \infty)$. The Rayleigh waves occur in the half-space only if the period of the effective stationary load in z is greater than $2\pi/\gamma\ (\xi < \gamma)$.

If the problem of stationary diffraction by a cavity in a half-space is considered, say, for harmonic waves whose potentials are given by the formulas

$$\Phi_k{}^0 = \exp(i\alpha_k((\mathbf{ex}) - c_k t)),\ \mathbf{x} = (x, y, z)$$

(k is a fixed subscript, and $\mathbf{e}$ is the unit vector of the wave propagation direction), then by introducing the potentials Φ_j of the reflected waves and setting $\varphi_j = \Phi_j{}^0 + \Phi_j$ by virtue of the linearity of the problem, we will arrive at the solution elucidated above for determining Φ_j. Since $\alpha_j < \gamma\ (j = 0, 1, 2),\ e_z = \cos A$ where A is the angle between the $\mathbf{e}$ vector and the OZ axis, we have $|e_z| \leqslant 1$, consequently $\xi = \alpha_j e_z,\ \xi < \gamma$. This means that Rayleigh waves also occur in the diffraction of periodic waves by a cylindrical cavity in a half-space.

If loads aperiodic in z are considered, this same problem occurs in the space of Fourier transforms in z. It can be shown that a two-dimensional surface exists in the space of variables (ξ, η) on which conditions (1.11) and (1.12) are satisfied and integration should be performed over this surface.

REFERENCES

1. MORSE P.M. and FESHBACH H., Methods of Theoretical Physics. /Russian translation/, 2, IIL, Moscow, 1960.
2. FRANK P. and MISES R., Differential and Integral Equations of Mathematical Physics /Russian translation/, Pt.2, Gostekhizdat, Moscow-Leningrad, 1973.
3. GUZ A.N. and GOLOVCHAN V.T., Diffraction of Elastic Waves in Multiconnected Bodies, Naukova Dumka, Kiev, 1972.
4. Handbook on Special Functions with Formulas, Graphs and Mathematical Tables, Nauka, Moscow, 1979.
5. ALEKSEYEVA L.A., On the vibrations of an elastic half-plane subjected to a stationary cylindrical wave source, Izv. Akad. Nauk Kaz. SSR, Ser. Fiz.-Mat., 5, 1983.
6. PRUDNIKOV A.P., BRYCHKOV YU.A. and MARICHEV O.I., Integrals and Series. Special Functions, Nauka, Moscow, 1983.
7. KANTOROVICH L.V. and KRYLOV V.I., Approximate Methods of Higher Analysis, Fizmatgiz, Moscow-Leningrad, 1962.
8. ALEKSEYEVA L.A., On the vibrations of an elastic half-plane weakened by a circular cavity Izv. Akad. Nauk Kaz. SSSR, SEr. Fiz.,-Mat., 1, 1984.

Translated by M.D.F.